2020年9月30日 10:11

MATH 2020 Tutorial 3 [Polar coordinates]

① $(x, y) \rightarrow (r, \theta)$

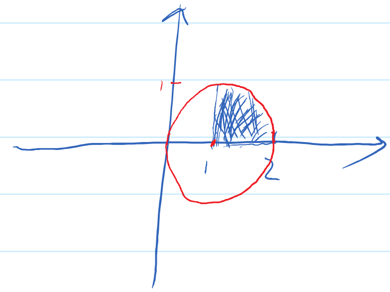
$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\, dy\, dx$$

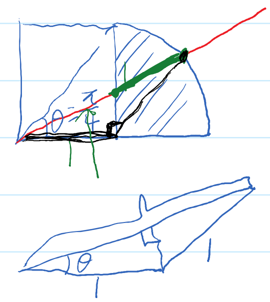
$y = \sqrt{2x-x^2}$

$y^2 = 2x - x^2$

$y^2 + x^2 - 2x = 0$

$y^2 + (x-1)^2 = 1$





$$= \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r^4}\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r^3}\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} -\frac{1}{2}\cdot\frac{1}{r^2} \Big|_{\frac{1}{\cos\theta}}^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} -\boxed{\frac{1}{8\cos^2\theta}} + \frac{\cos^2\theta}{2}\, d\theta$$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

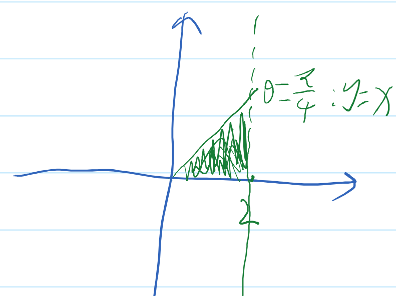
$$= -\frac{1}{8}\tan\theta \Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{4}\, d\theta$$

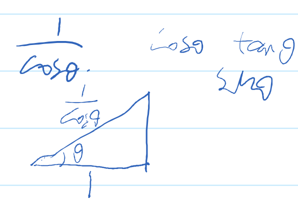
$$= -\frac{1}{8} + \left(\frac{\sin 2\theta}{8} + \frac{\theta}{4}\right)\Big|_0^{\frac{\pi}{4}} = -\frac{1}{8} + \frac{1}{8} + \frac{\pi}{16} = \frac{\pi}{16}$$

② $(r, \theta) \longrightarrow (x, y)$

$$\int_0^{\frac{\pi}{4}} \int_0^{2\sec\theta} \underbrace{r^5 \sin^2\theta\, dr\, d\theta}_{= r^2 \cdot (r\sin\theta)^2 (r\, dr\, d\theta)}$$

$$= \int_0^2 \int_0^x (x^2+y^2)\, y^2\, dy\, dx.$$





③ Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Ans: $r = 1 + \cos\theta$

Ans: $r = 1+\cos\theta$

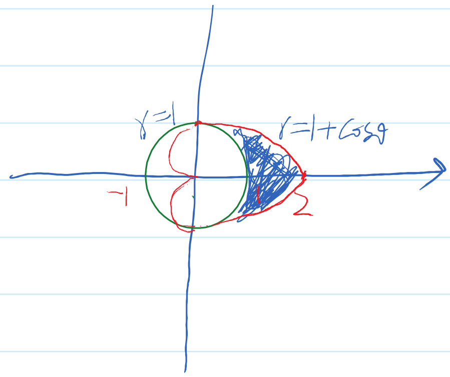
$\theta = 0, \quad r = 2$

$0 \to \theta \to \frac{\pi}{2}, \quad 1 \to \cos\theta \to 0$

$\frac{\pi}{2} \to \theta \to \pi, \quad 0 \to \cos\theta \to -1$

$\pi \to \theta \to \frac{3}{2}\pi, \quad -1 \to \cos\theta \to 0$

$\frac{3}{2}\pi \to \theta \to 2\pi, \quad 0 \to \cos\theta \to 1$



Area is $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} r\,dr\,d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big|_{1}^{1+\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(\cancel{1} + 2\cos\theta + \cos^2\theta \cancel{-1})\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta + \frac{\cos^2\theta}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta + \frac{\cos 2\theta + 1}{4}\, d\theta$$

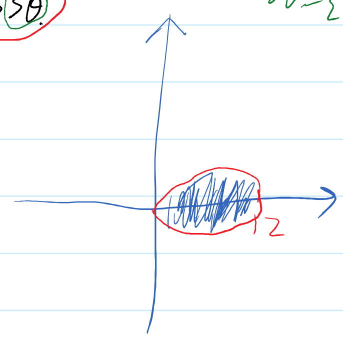
$$= \sin\theta + \frac{\sin 2\theta}{8} + \frac{\theta}{4} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2 + \frac{\pi}{4}$$

④ Find the area enclosed by one leaf of $r = 12\cos 3\theta$.

$3\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$

Ans: $0 \to \theta \to \frac{\pi}{6}, \quad 12 \to r \to 0$

$-\frac{\pi}{6} \to \theta \to 0 \qquad 0 \to r \to 12$



Area $= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_{0}^{12\cos 3\theta} r\,dr\,d\theta.$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} r^2 \Big|_{0}^{12\cos 3\theta} d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 72\cos^2 3\theta\, d\theta.$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 36(\cos(6\theta)+1)\, d\theta = 6\sin 6\theta + 36\theta \Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = 12\pi.$$

⑤ $\int_{0}^{\frac{\pi}{4}} \int_{0}^{\sec y} x^2 \cos y\, dx\, dy$

$$= \int_{0}^{\frac{\pi}{4}} \frac{x^3}{3}\cos y \Big|_{0}^{\sec y} dy = \int_{0}^{\frac{\pi}{4}} \frac{\sec^3 y}{3} \cdot \cos y\, dy$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{1}{3\cos^2 y}\, dy = \frac{1}{3}\tan y \Big|_{0}^{\frac{\pi}{4}} = \frac{1}{3}$$

$\theta = \frac{\pi}{4}: y = x$

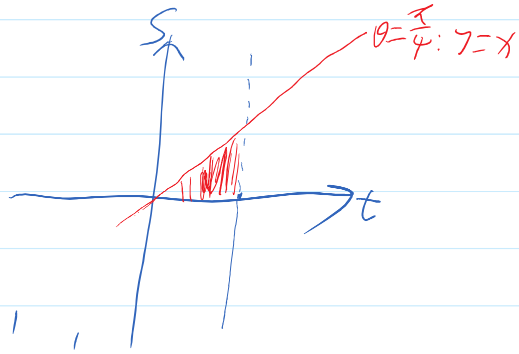
$= \int_0 \frac{\ }{3\cos^2 y} dy = \frac{\ }{3} \tan y \Big|_0^{\frac{\pi}{4}} = \frac{\ }{3}$

Another method: set $x = r$, $y = \theta$

$$\int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \underbrace{r^2 \cos\theta \, dr d\theta}_{= r\cos\theta \cdot r dr d\theta}$$

$$= \int_0^1 \boxed{\int_0^t t \, ds} \, dt$$

$$= \int_0^1 ts \Big|_0^t dt = \int_0^1 t^2 dt = \frac{1}{3} t^3 \Big|_0^1 = \frac{1}{3}$$



(6) Discuss the existence of the improper integral.

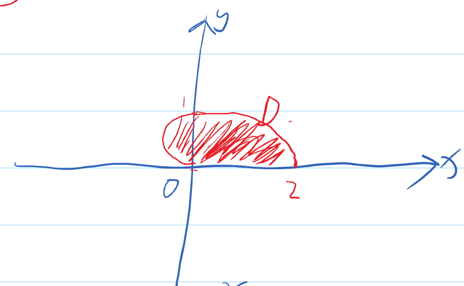
$$\iint_D \frac{y}{[x^2+y^2]^{\frac{3}{2}}} \to \infty \text{ as } r \to 0.$$

where D is the region enclosed by $r = 1 + \cos\theta$ and $y \geq 0$.

Ans: $\iint_D \frac{y}{[x^2+y^2]^{\frac{3}{2}}} dy dx$

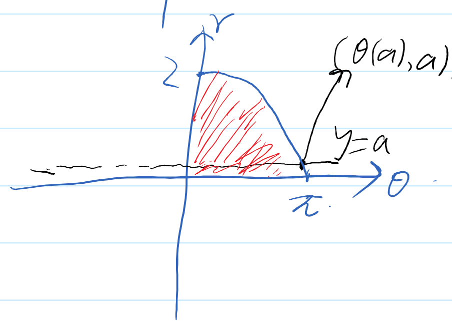
$$= \int_0^{\pi} \int_0^{1+\cos\theta} \frac{r \sin\theta}{r^3} r dr d\theta$$

$$= \int_0^{\pi} \int_0^{1+\cos\theta} \frac{\sin\theta}{r} dr d\theta. \qquad r \to 0$$



Let $D(a) = \{ (r, \theta) : 0 \leq \theta \leq \theta(a), \ a \leq r \leq 1 + \cos\theta \}$

when $a \to 0$, $\theta(a) \to \pi$, $D(a) \to D$



$$\iint_{D_a} \frac{\sin\theta}{r} dr d\theta$$

$$= \int_0^{\theta(a)} \int_a^{1+\cos\theta} \frac{\sin\theta}{r} dr d\theta$$

$$= \int_0^{\theta(a)} \sin\theta \cdot \lg r \Big|_a^{1+\cos\theta} d\theta$$

$$= \int_0^{\theta(a)} \sin\theta \left[\overset{\text{welldefined}}{\lg(1+\cos\theta)} - \overset{\to +\infty}{\lg a} \right] d\theta$$

$-\infty$ $\int_0^{\theta(a)} \sin\theta \cdot \lg a \, d\theta = -\lg a \cdot \cos\theta \Big|_0^{\theta(a)} = \underbrace{-\overset{-\infty}{\lg a} \overset{-2}{\left[\cos\theta(a) - 1 \right]}}_{\to -\infty}$ $\qquad a \to 0, \theta(a) \to \pi$

Ok. $\int_0^{\theta(a)} \sin\theta \lg(1+\cos\theta) d\theta \qquad (t = \cos\theta)$

$\int_{\cos\theta(a)}$

Ok. $\int_0^{a} \sin\theta \, \lg(1+\cos\theta)\, d\theta$ $(t=\cos\theta)$

$$= \int_1^{\cos\theta(a)} -\lg(1+t)\,dt = \int_{\cos\theta(a)}^{1} \lg(1+t)\,dt$$

$$= t\cdot \lg(1+t)\Big|_{\cos\theta(a)}^{1} - \int_{\cos\theta(a)}^{1} 1-\frac{1}{1+t}\,dt$$

$$= \lg 2 - \cos\theta(a)\lg(1+\cos\theta(a)) - (1-\cos\theta(a)) + [\lg 2 - \lg(1+\cos\theta(a))]$$

$$= 2\lg 2 - 1 + \cos\theta(a) - \underbrace{(\cos\theta(a)+1)}_{\to 0}\underbrace{\lg(\cos\theta(a)+1)}_{\to -\infty} \quad a\to 0,\ \theta(a)\to\pi,\ \cos\theta(a)\to -1$$

$\to 0$

MATH 2020 Tutorial 3

① $(x,y)\to(r,\theta)$

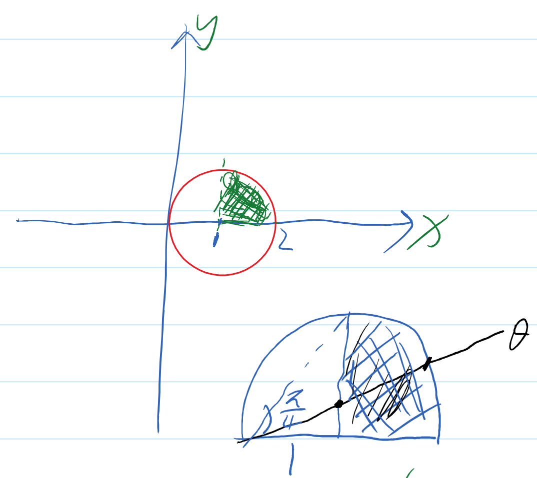
$$\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\,dy\,dx$$

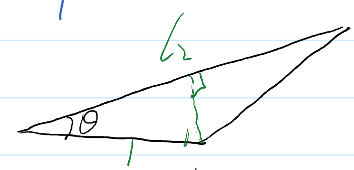
$y=\sqrt{2x-x^2}$

$y^2=2x-x^2$

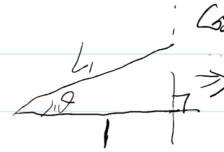
$y^2+x^2-2x=0$

$y^2+(x-1)^2=1$





$\cos\theta=\frac{1}{l_1}$



$\Rightarrow l_1=\frac{1}{\cos\theta}$

$$= \int_0^{\frac{\pi}{4}}\int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r^4}\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}}\int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r^3}\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} -\frac{1}{2}\cdot\frac{1}{r^2}\Big|_{\frac{1}{\cos\theta}}^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} -\frac{1}{8\cos^2\theta} + \frac{\cos^2\theta}{2}\,d\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$= -\frac{1}{8}\tan\theta\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta+1}{4}\,d\theta$$

$$= -\frac{1}{8}\tan\theta\Big|_0^{\frac{\pi}{4}} + \frac{\sin 2\theta}{8} + \frac{\theta}{4}\Big|_0^{\frac{\pi}{4}}$$

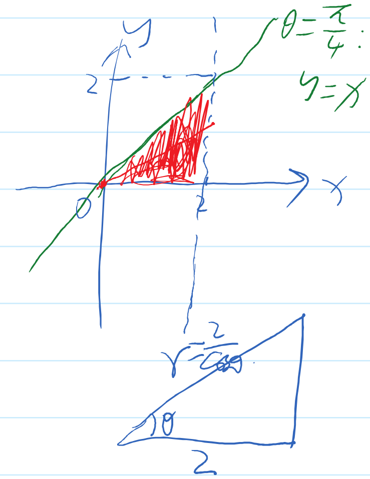
$= -\frac{1}{8} + \frac{1}{8} + \frac{\pi}{16} = \frac{\pi}{16}.$

② $(r,\theta) \to (x,y)$

$$\int_0^{\frac{\pi}{4}} \int_0^{2\sec\theta} \underbrace{r^5 \sin^2\theta \, dr d\theta}_{(r\sin\theta)^2 \cdot r^2 \cdot r dr d\theta}$$

$r = 2\sec\theta = \frac{2}{\cos\theta}$.

$$= \int_0^2 \int_0^x y^2(x^2+y^2)\, dy dx.$$



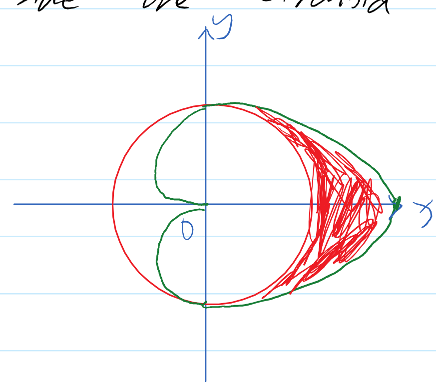
③ Find the area of the region that lies inside the cardioid $r = 1+\cos\theta$ and outside the circle $r = 1$.

Ans: $0 \to \theta \to \frac{\pi}{2}$, $1 \to \cos\theta \to 0$, $2 \to r \to 1$

$\frac{\pi}{2} \to \theta \to \pi$, $0 \to \cos\theta \to -1$, $1 \to r \to 0$

$\pi \to \theta \to \frac{3\pi}{2}$, $-1 \to \cos\theta \to 0$, $0 \to r \to 1$

$\frac{3\pi}{2} \to \theta \to 2\pi$, $0 \to \cos\theta \to 1$, $1 \to r \to 2$.



$$\text{Area} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{1+\cos\theta} r\, dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big|_1^{1+\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(\cancel{1} + 2\cos\theta + \cos^2\theta \cancel{-1})\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta + \frac{\cos^2\theta}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta + \frac{\cos 2\theta + 1}{4}\, d\theta$$

$$= \sin\theta + \frac{\sin 2\theta}{8} + \frac{\theta}{4} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= (1+1) + 0 + \frac{1}{4}\cdot\pi = 2 + \frac{\pi}{4}$$

④ Find the area enclosed by one leaf of $r = 12\cos 3\theta$.

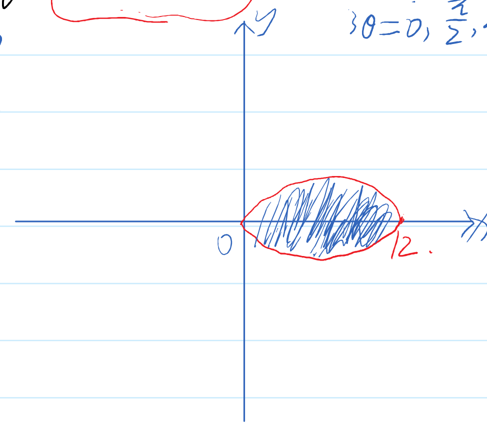
Ans: $0 \to \theta \to \frac{\pi}{?}$, $1 \to \cos 3\theta \to 0$, $12 \to r \to 0$

$3\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

(4) Find the area enclosed by one leaf of $r=12\cos 3\theta$.

Ans: $0\to\theta\to\frac{\pi}{6}$, $1\to\cos 3\theta\to 0$, $12\to r\to 0$

$-\frac{\pi}{6}\to\theta\to 0$, $0\to\cos 3\theta\to 1$, $0\to r\to 12$.



$$\text{Area}=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_0^{12\cos 3\theta} r\,dr\,d\theta.$$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}r^2\Big|_0^{12\cos 3\theta} d\theta.$$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 72\cos^2 3\theta\,d\theta.$$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \overset{36}{\cancel{72}}\cdot\frac{\cos 6\theta+1}{\cancel{2}}\,d\theta.$$

$$=6\sin 6\theta+36\theta\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}=36\cdot\left(\frac{\pi}{6}+\frac{\pi}{6}\right)=12\pi.$$

(5) $\int_0^{\frac{\pi}{4}}\int_0^{\sec y} x^2\cos y\,dx\,dy$

$$=\int_0^{\frac{\pi}{4}} \frac{1}{3}x^3\cos y\Big|_0^{\sec y}dy=\int_0^{\frac{\pi}{4}}\frac{1}{3}\sec^3 y\cos y\,dy$$

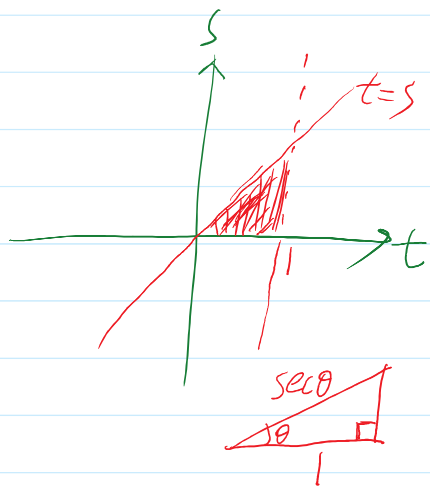
$$=\int_0^{\frac{\pi}{4}}\frac{1}{3\cos^2 y}dy=\frac{1}{3}\tan y\Big|_0^{\frac{\pi}{4}}=\frac{1}{3}$$

Another method: set $y=\theta$, $x=r$.

$$\int_0^{\frac{\pi}{4}}\int_0^{\sec\theta}\underbrace{r^2\cos\theta\,dr\,d\theta}_{=r\cos\theta\cdot r\,dr\,d\theta}$$

$$=\int_0^1\int_0^t t\,ds\,dt$$

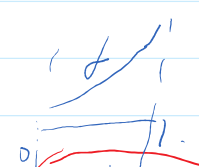
$$=\int_0^1 ts\Big|_0^t dt=\int_0^1 t^2dt=\frac{1}{3}$$



(6) Discuss the existence of the improper integral.

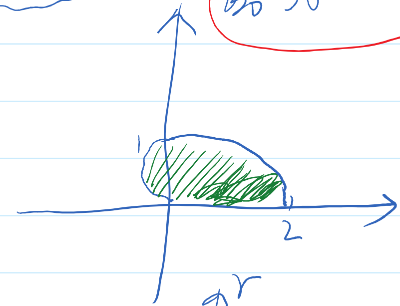
$$\iint_D \frac{y}{[x^2+y^2]^{\frac{3}{2}}} \to\infty \text{ as } r\to 0.$$

where D is the region enclosed by $r=1+\cos\theta$, $y\geq 0$. $\lim\int^{1-\varepsilon} f\,dx$



$\iint_D \quad [x^2+y^2]^{\frac{3}{2}}$

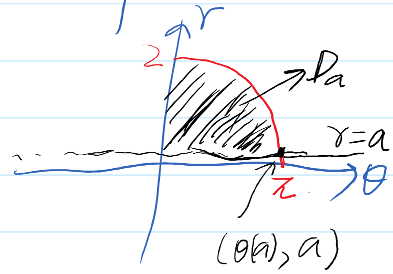
where D is the region enclosed by $r = 1+\cos\theta$, $y \geq 0$. $\quad \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} f\,dx$



Ans. $\displaystyle \int_0^{\pi}\int_0^{1+\cos\theta} \frac{r\sin\theta}{r^3}\, r\,dr\,d\theta$

$\displaystyle = \int_0^{\pi}\int_0^{1+\cos\theta} \frac{\sin\theta}{r}\,dr\,d\theta$

$\displaystyle \iint_{D_a} \frac{\sin\theta}{r}\,dr\,d\theta$



$a \to 0$, $\theta(a) \to \pi$

$\displaystyle = \int_0^{\theta(a)}\int_a^{1+\cos\theta} \frac{\sin\theta}{r}\,dr\,d\theta$

$\displaystyle = \int_0^{\theta(a)} \sin\theta \cdot \lg r \Big|_a^{1+\cos\theta}\, d\theta$

$\displaystyle = \int_0^{\theta(a)} \underbrace{\sin\theta \cdot \lg(1+\cos\theta)}_{\text{I well-defined}} - \underbrace{\sin\theta \cdot \lg a}_{\text{II} \to -\infty}\, d\theta$

$\displaystyle \text{II} = \int_0^{\theta(a)} \sin\theta \lg a\, d\theta = -\cos\theta \lg a \Big|_0^{\theta(a)} = (\underbrace{-\cos\theta(a) + 1}_{\to 2}) \underbrace{\lg a}_{\to -\infty}$ $\to -\infty$ as $a \to 0$.

$a \to 0$, $\theta(a) \to \pi$, $\cos\theta(a) \to -1$, $\lg a \to -\infty$

$\displaystyle \text{I} = \int_0^{\theta(a)} \sin\theta \lg(1+\cos\theta)\, d\theta \quad (\text{set } t = \cos\theta)$

$\displaystyle = \int_1^{\cos\theta(a)} -\lg(1+t)\, dt = \int_{\cos\theta(a)}^{1} \lg(1+t)\,dt$

$\displaystyle = t\lg(1+t) \Big|_{\cos\theta(a)}^{1} - \int_{\cos\theta(a)}^{1} \frac{1+t-1}{1+t}\, dt$

$= \lg 2 - \cos\theta(a) \lg[1+\cos\theta(a)] - (1-\cos\theta(a)) + \lg 2 - \lg[1+\cos\theta(a)]$

$= 2\lg 2 - 1 + \underset{\downarrow\, -1}{\cos\theta(a)} - \underset{\downarrow\, 0}{[1+\cos\theta(a)]}\, \underset{\downarrow\, -\infty}{\lg[1+\cos\theta(a)]}$

$a \to 0$, $\theta(a) \to \pi$, $\cos\theta(a) \to -1$

$\displaystyle \lim_{x \to 0} x \cdot \lg x = 0$